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Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations

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ABSTRACT

We discuss the convergence properties of spline collocation and iterated collocation methods for a weakly singular Volterra integral equation associated with certain heat conduction problems. This work completes the previous studies of numerical methods for this type of equations with noncompact kernel. In particular, a global convergence result is obtained and it is shown that discrete superconvergence can be achieved with the iterated collocation if the exact solution belongs to some appropriate spaces. Some numerical examples illustrate the theoretical results.

1. Introduction

The Volterra integral equation

$$y(t) = \int_0^t K(t,s)p(t,s)y(s)\mathrm{d}s + g(t), \quad t \in I = (0,T], \tag{1.1}$$

$$p(t,s) := \frac{s^{\mu-1}}{t^{\mu}}, \tag{1.2}$$

where $\mu > 0$, $K(t,s)$ is a smooth function and g is a given function, can arise, e.g., in heat conduction problems with mixed boundary conditions ([2,10]). We start by giving the following lemma which summarizes some analytical results for (1.1) in the case $K(t,s) = 1$.

Lemma 1.1. (a) [13] *Let* $\mu > 1$ *in* (1.2). *If the function* g *belongs to* $C^m[0,T]$ *then the integral equation*

$$y(t) = \int_0^t p(t,s)y(s)\mathrm{d}s + g(t), \quad t \in (0,T], \tag{1.3}$$

possesses a unique solution $y \in C^m[0,T]$.
(b) [17] *In the above case* (a), *the unique solution* $y \in C^m[0,T]$ *is given by:*

$$y(t) = g(t) + t^{1-\mu}\int_0^t s^{\mu-2}g(s)\mathrm{d}s. \tag{1.4}$$

(c) [17] *However, if* $0 < \mu \le 1$ *and* $g \in C^1[0,T]$ *(with* $g(0) = 0$ *if* $\mu = 1$*), then* (1.3) *has a family of solutions in* $C[0,T]$ *of which only one has* C^1 *continuity.*

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We note that Eq. (1.3) differs substantially from the well-known case of Abel-type equations

$$y(t) = (Ay)(t) + g(t), \tag{1.5}$$

where

$$(Ay)(t) := \int_0^t (t-s)^{-\alpha} y(s)\mathrm{d}s, \quad 0 < \alpha < 1. \tag{1.6}$$

Let us associate with Eq. (1.3) the integral operator

$$(Ly)(t) := \int_0^t \frac{s^{\mu-1}}{t^{\mu}} y(s)\mathrm{d}s, \quad (\mu > 1). \tag{1.7}$$

An important difference between the two kinds of integral operators defined above is illustrated as follows. Defining $\phi_r(s) = s^r$, $r \geq 0$, $s \geq 0$, it is easily obtained that

$$(L\phi_r)(t) = \int_0^t \frac{s^{\mu-1+r}}{t^{\mu}} \mathrm{d}s = \lambda_r \phi_r(t), \tag{1.8}$$

with

$$\lambda_r = \int_0^1 s^{\mu-1+r} \mathrm{d}s = \frac{1}{\mu + r}.$$

Since $r \geq 0$ then $\lambda_r \in (0, \infty)$. In other words, the spectrum of operator L is not discrete, therefore L is not a compact operator in $C(I)$. This is in contrast with the Abel operator theory (see e.g. [16,4]). Therefore the classical analytical and numerical theory does not apply to the above Eq. (1.3).

The numerical solution of integral equations with an Abel-type kernel of the form $p(t, s, y(s))(t-s)^{-\alpha}$, $0 < \alpha < 1$, where $p(t, s, y)$ is sufficiently smooth, has been considered by many authors. For these equations, a smooth forcing function leads to a solution which has typically unbounded derivatives at $t = 0$. As a consequence of this nonsmooth behaviour, if uniform meshes are used then the convergence order of polynomial spline collocation methods is only $1 - \alpha$, independently of the degree of the polynomials (cf. [6]). In order to recover the optimal convergence orders one has to use suitable graded meshes ([3,8,22]). Alternatively, one may keep the uniform meshes but then use nonpolynomial spline approximating functions reflecting the singularity ([5,9,21]); variable transformations followed by standard methods have also been considered by several authors (e.g. [1,14,15,19,20]). For an extensive list of references on these and other approaches see [4]. A particular nonlinear Abel-type equation, where the function $p(t, s, y)$ in its kernel is nonsmooth, has been considered in [12].

Eq. (1.3) has been the subject of several works for the case when $\mu > 1$. Certain classes of product integration methods based on Newton–Cotes rules were studied in [10]; Diogo et al. [13] considered a fourth-order Hermite-type collocation method and Lima and Diogo [18] developed an extrapolation algorithm, based on Euler's method.

The present work is concerned with collocation methods based on general polynomial spline functions. In Section 2 it is shown that piecewise polynomials of degree $m - 1$ on uniform meshes yield global convergence of order m. The performance of the corresponding iterated collocation solution is also investigated and it is shown that, under certain conditions, a remarkable improvement in the convergence orders at the mesh points can be obtained by using the Gauss points as collocation points. A theoretical proof of this result is given in Section 4 and the paper concludes with a sample of numerical examples illustrating the performance of the methods. The above properties are in sharp contrast with the situation for Abel-type equations for which no superconvergence properties can be obtained (see [7,6]).

2. A global convergence result

Let us consider equation (1.3), that is, (1.1) with $K(t, s) = 1$, and $\mu > 1$. In [17] some more general cases of K were considered, namely $K(t, t) = 1$ and $K(t, s)$ =constant. There some results on existence and uniqueness of solution in the spaces $C^m[0, T]$ were presented and some regularity estimates were derived. In principle it is possible to consider other generalizations, like the case when $K(t, s) = h(t)$. However a resolvent formula like (1.4) might not be possible to obtain. Different approaches to prove convergence of numerical methods are the subject of further investigation.

We follow the notations of [6]. Given the following partition of the interval I

$$\left\{t_j = jh, \quad 0 \leq j \leq N;\, Nh = T\right\},$$

let $\sigma_0 := [t_0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$, $1 \leq n \leq N - 1$ and define $Z_N := \{t_n : n = 1, \ldots, N-1\}$, $\overline{Z}_N := Z_N \bigcup T$. Furthermore, let π_{m-1} denote the space of polynomials of degree $m - 1$. The exact solution of (1.3) will be approximated in the piecewise polynomial space

$$S_{m-1}^{-1}(Z_N) := \{u : u|_{\sigma_n} =: u_n \in \pi_{m-1}, \quad 0 \leq n \leq N-1\}, \tag{2.1}$$

whose elements, in general, will possess jump discontinuities at their knots Z_N. Consider the following finite subset of I

$$X(N) = \bigcup_{n=0}^{N-1} X_n, \tag{2.2}$$

with

$$X_n := \{t_{nj} := t_n + c_j h : 0 \le c_1 < \cdots < c_m \le 1\}. \tag{2.3}$$

The approximate solution $u \in S_{m-1}^{-1}(Z_N)$ will be required to satisfy the original Eq. (1.3) on $X(N)$ (set of collocation points). We thus have the following collocation equation

$$u(t) = g(t) + \int_0^t p(t,s)u(s)\mathrm{d}s, \quad t \in X(N), \tag{2.4}$$

which can be rewritten as

$$u_n(t_{nj}) = g(t_{nj}) + \int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} u_n(t_n+\tau h)\mathrm{d}\tau + \sum_{i=0}^{n-1}\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} u_i(t_i+\tau h)\mathrm{d}\tau,$$
$$j = 1,\ldots,m \ (n = 0,\ldots,N-1). \tag{2.5}$$

Let λ_l be the canonical Lagrange polynomials associated with the collocation parameters, defined by

$$\lambda_l(\tau) := \prod_{\substack{i=1\\ i\neq l}}^{m} \frac{\tau - c_i}{c_l - c_i}. \tag{2.6}$$

If on each subinterval σ_n u is given by its Lagrange formula

$$u_n(t_n+\tau h) := \sum_{l=1}^{m} \lambda_l(\tau)u_{nl}, \quad t_n + \tau h \in \sigma_n, \tag{2.7}$$

where $u_{nl} := u_n(t_n + c_l h)$, then (2.5) represents a sequence of N linear systems in the unknowns $(u_{n1},\ldots,u_{nm})^T$, $0 \le n \le N-1$:

$$u_{nj} = g(t_{nj}) + \sum_{l=1}^{m}\left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}}\lambda_l(\tau)\mathrm{d}\tau\right)u_{nl} + \sum_{i=0}^{n-1}\sum_{l=1}^{m}\left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}}\lambda_l(\tau)\mathrm{d}\tau\right)u_{il}. \tag{2.8}$$

We see that the integrals in (2.8) can be evaluated analytically.

Let $e := y - u$ and denote its restriction to the subinterval σ_n by e_n. We define

$$\|e\|_\infty := \sup\{|e_n(t)| : t \in \sigma_n, \quad n = 0,\ldots,N-1\}.$$

We have the following global convergence result.

Theorem 2.1. *Suppose $\mu > 1$ in (1.2). Let $u \in S_{m-1}^{(-1)}(Z_N)$ denote the collocation approximation to the solution of the integral equation (1.3) and assume that $g \in C^m(I)$. Then, for every choice of the collocation parameters $\{c_j\}$, with $0 \le c_1 < \cdots < c_m \le 1$, we have*

$$\|e\|_\infty = O(h^m), \quad (\text{as } h \downarrow 0, Nh = T). \tag{2.9}$$

Proof. From Lemma 1.1 it follows that (1.3) has a unique solution $y \in C^m(I)$. Then we may write

$$y(t_n + vh) = \sum_{l=1}^{m} y(t_n + c_l h)\lambda_l(v) + h^m E_n(y; v), \quad t_n + vh \in \sigma_n, \tag{2.10}$$

where the λ_l are defined by (2.6) and

$$E_n(y; v) = \frac{y^{(m)}(\xi_n)}{m!}\prod_{j=1}^{m}(v - c_j), \quad t_n < \xi_n < t_n + vh. \tag{2.11}$$

Combining (2.10) and (2.7), results in the following equation for the restriction of the error function to the interval σ_n, $(n = 0, 1, \ldots, N-1)$,

$$e_n(t_n + vh) = \sum_{k=1}^{m} \lambda_k(v)(y(t_n + c_k h) - u_{nk}) + h^m E_n(y; v). \tag{2.12}$$

Let

$$\Gamma_m := \max\left\{\sum_{k=1}^{m} |\lambda_k(v)| : 0 \le v \le 1\right\}$$

denote the Lebesgue constant associated with the collocation parameters $\{c_j\}$ and set

$$E := \max\{|e_n(t_{nk})|, \quad k = 1,\ldots,m, n = 0,\ldots,N-1\}.$$

Then we have, by (2.12),

$$|e_n(t_n+vh)| \le \sum_{k=1}^{m} |e_n(t_{nl})||\lambda_k(v)| + h^m|E_n(y;v)| \le E\Gamma_m + C_m h^m, \tag{2.13}$$

with $C_m := \max_{t\in I} |y^{(m)}(t)/m! \prod_{j=1}^{m}(t-c_j)|$. If $E = O(h^m)$ then from (2.13) we obtain, for some constant C independent of h,

$$|e_n(t_n+vh)| \le Ch^m, \tag{2.14}$$

which is equivalent to (2.9).

In order to prove that $E = O(h^m)$, it will be convenient to rewrite the collocation Eq. (2.4) in the form

$$u(t) = g(t) + \int_0^t p(t,s)u(s)\mathrm{d}s - \delta(t), \quad t \in I, \tag{2.15}$$

where δ is a suitable function which is zero at the collocation points. Let us define

$$\Upsilon(t) := \int_0^t p(t,s)u(s)\mathrm{d}s = \int_0^t \frac{s^{\mu-1}}{t^\mu}u(s)\mathrm{d}s.$$

Since $u \in S_{m-1}^{-1}(Z_N)$, that is, u is a piecewise polynomial function of degree $m-1$, then we also have that $\Upsilon \in S_{m-1}^{-1}(Z_N)$. Moreover, from (2.15) we can conclude that δ will be smooth on each subinterval, with the degree of smoothness given by that of g.

Subtracting (2.15) from (1.3) gives the following second kind Volterra integral equation for the error function

$$e(t) = \delta(t) + \int_0^t p(t,s)e(s)\mathrm{d}s, \quad t \in I. \tag{2.16}$$

If $c_m = 1$ then δ will be continuous and it follows from Lemma 1.1 that the solution of the above equation is given by

$$e(t) = \delta(t) + \int_0^t R(t,s)\delta(s)\mathrm{d}s, \quad t \in I, \tag{2.17}$$

with

$$R(t,s) = s^{\mu-2}/t^{\mu-1}. \tag{2.18}$$

If $c_m < 1$ then δ may possess jump discontinuities at the mesh points Z_N and it can be shown that the formula (2.17) still applies. Setting $t = t_{nj}$ in (2.17) and making appropriate changes of variables yields

$$e(t_{nj}) = \delta(t_{nj}) + \sum_{i=0}^{n-1} h\int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\delta(t_i+\tau h)\mathrm{d}\tau + h\int_0^{c_j} \frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\delta(t_n+\tau h)\mathrm{d}\tau, \quad j = 1,\dots,m. \tag{2.19}$$

Using the Lagrange formula for δ based on the m-point abscissas t_{il}, we obtain

$$\int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\delta(t_i+\tau h)\mathrm{d}\tau = \int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\left(\sum_{l=1}^{m}\lambda_l(\tau)\delta(t_i+c_l h) + h^m E_i(\delta;\tau)\right)\mathrm{d}\tau, \tag{2.20}$$

$$\int_0^{c_j} \frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\delta(t_n+\tau h)\mathrm{d}\tau = \int_0^{c_j} \frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\left(\sum_{l=1}^{m}\lambda_l(\tau)\delta(t_n+c_l h) + h^m E_n(\delta;\tau)\right)\mathrm{d}\tau. \tag{2.21}$$

Above the λ_l are the canonical Lagrange polynomials (2.6) and

$$E_i(\delta;\tau) := \frac{\delta^{(m)}(\xi_i)}{m!}\prod_{k=1}^{m}(\tau-c_k), \quad t_i < \xi_i < t_i+\tau h.$$

Substituting (2.20) and (2.21) into (2.19), and since $\delta(t)$ vanishes at the collocation points, we are led to

$$e(t_{nj}) = \sum_{i=0}^{n-1} h\int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}h^m E_i(\delta;\tau)\mathrm{d}\tau + h\int_0^{c_j}\frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}h^m E_n(\delta;\tau)\mathrm{d}\tau. \tag{2.22}$$

Applying modulus gives

$$|e(t_{nj})| \le h^m\bar{E}\left(\sum_{i=0}^{n-1} h\int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\mathrm{d}\tau + h\int_0^{c_j}\frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}}\mathrm{d}\tau\right), \tag{2.23}$$

where $\bar{E} := \sup\{|E_i(\delta;\tau)|, i = 0, 1, \ldots, N-1\}$. Using the fact that

$$\sum_{i=0}^{n-1} h \int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}} \mathrm{d}\tau + h \int_0^{c_j} \frac{(t_n+\tau h)^{\mu-2}}{(t_{nj})^{\mu-1}} \mathrm{d}\tau = \int_0^{t_{nj}} R(t_{nj},\tau)\mathrm{d}\tau = \frac{1}{\mu-1} \tag{2.24}$$

yields

$$|e(t_{nj})| \le \frac{h^m}{\mu-1}\bar{E} = O(h^m), \quad t_{nj} \in X_N, \tag{2.25}$$

which, together with (2.13), completes the proof of the theorem. ⋄

3. Superconvergence

In [11] the discrete superconvergence properties of the collocation solutions for Eq. (1.3) have been investigated. By this we mean the possibility of attaining a higher-order $p > m$ at the mesh points, by a special choice of the collocation points. The following lemma was used in the superconvergence analysis of collocation (with $q = 1$ for Radau II points and $q = 2$ for Lobatto points) and will be needed here with $q = 0$.

Lemma 3.1. *Let us assume that $y \in C^{2m-q}([0, T])$, $q \in \{0, 1, 2\}$, and*

$$y^{(m)}(0) = y^{(m+1)}(0) = \ldots y^{(2m-q-1)}(0) = 0. \tag{3.1}$$

Then we have for the collocation error on the first subinterval

$$\max_{1\le j\le m} |e_0(t_{0j})| = O(h^{2m-q}). \tag{3.2}$$

We see that the above conditions (3.1) are somewhat restrictive in the sense that they will not be satisfied in many problems. However, if μ is sufficiently high, it is possible to transform the original equation into a new equation to which the above lemma can be applied. Let $\beta \ge 0$ be such that $\mu - \beta > 1$. Multiplying both sides of (1.3) by t^β and defining

$$\bar{y}(t) := t^\beta y(t), \quad \bar{g}(t) := t^\beta g(t), \tag{3.3}$$

we obtain the equation

$$\bar{y}(t) - \int_0^t \left(\frac{s}{t}\right)^{\mu-\beta} \frac{1}{s}\, \bar{y}(s)\mathrm{d}s = \bar{g}(t), \tag{3.4}$$

which is equivalent to (1.3) for $t > 0$. The idea is to choose β such that $\bar{y}(t) = t^\beta y(t)$ will be sufficiently smooth and satisfy the required conditions (3.1) for superconvergence. We recall some definitions and notations from [18].

Definition 3.1. For $T > 0$ and m a nonnegative integer, $V_m[0, T]$ denotes the normed space of the real valued functions f such that $f \in C^m[0, T]$ with

$$\|f\|_m := \max_{0\le j\le m} \max_{t\in[0,T]} \left|f^{(j)}(t)\right|. \tag{3.5}$$

Definition 3.2. For $T > 0$, $\beta \ge 0$ and $m \ge 0$, $V_{m,\beta}[0, T]$ denotes the normed space of real valued functions f such that $t^\beta f \in V_m[0, T]$ with

$$\|f\|_{m,\beta} := \|t^\beta f\|_m = \max_{0\le j\le m} \max_{t\in[0,T]} \left|\frac{\mathrm{d}^j}{\mathrm{d}t^j}(t^\beta f(t))\right|. \tag{3.6}$$

Definition 3.3. Let $m \ge 1$. Then $f \in V_m^0[0, T]$ if and only if $f \in V_m[0, T]$ and $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$. If $m = 0$, we have $V_0^0[0, T] = V_0[0, T]$.

Definition 3.4. Let $m \ge 1$ and $\beta \ge 0$. Then $f \in V_{m,\beta}^0[0, T]$ if and only if $t^\beta f \in V_m^0[0, T]$. If $m = 0$, we have $V_{0,\beta}^0[0, T] = V_{0,\beta}[0, T]$.

Further details and results concerning existence and uniqueness of solution in these spaces can be found in [18]. The following result is a corollary of Lemma 3.1.

Lemma 3.2. *If the solution y of Eq. (1.3) belongs to the space $V_{2m-q,\beta}^0[0, T]$, for some $0 \le \beta < \mu - 1$, then the collocation error on the first subinterval is of order $2m - q$.*

Proof. Since the solution $\bar{y}(t) = t^{\beta} y(t)$ of (3.4) will belong to $V^0_{2m-q}[0, T]$, then, according to Definition 3.3, $\bar{y}$ satisfies condition (3.1) and the result follows from Lemma 3.1. $\diamond$

In [11] it was proved that superconvergence of order $2m - 1$ could be attained with the Radau II points if the solution y was in the continuity class C^{2m-1} and satisfied (3.1) with $q = 1$. Using the spaces above and taking into account Lemma 3.2, the following result can be proved.

Lemma 3.3. *Let the collocation parameters $\{c_j\}$ be the Radau* II *points for $(0, 1]$, that is, the zeros of $P_{m-1}(2s-1) - P_m(2s-1)$, where P_m is the m-th degree Legendre polynomial. If the solution y of Eq. (1.3) belongs to the space $V^0_{2m-1,\beta}[0, T]$, for some $0 \le \beta < \mu - 1$, then superconvergence of order $2m - 1$ occurs at the mesh points.*

4. Iterated collocation

Although with the Gauss–Legendre points (that is, the zeros of the shifted Legendre polynomial $P_m(2s - 1)$) superconvergence does not occur at the mesh points, we show here that a high local order can be attained if we consider the so-called iterated collocation.

The iterated approximation u^I associated with the collocation solution u is defined by

$$u^I(t) := g(t) + \int_0^t p(t, s)u(s)\mathrm{d}s, \quad t \in I. \tag{4.1}$$

Setting $t = t_n \in \bar{Z}_N$ in (4.1) and using the Lagrange formula (2.7) for u, yields

$$u^I(t_n) = g(t_n) + \sum_{i=0}^{n-1}\sum_{l=1}^{m}\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+1)^{\mu}}\lambda_l(\tau)\mathrm{d}\tau\, u_{il}. \tag{4.2}$$

Theorem 4.1. *Let $\mu > 1$ and $u \in S^{(-1)}_{m-1}(Z_N)$ denote the collocation approximation to the solution of the integral equation (1.3), defined by (2.4), where the collocation parameters $\{c_j\}$ are the Gauss points for $(0, 1)$, and let $g \in C^{2m}[0, T]$. Suppose that the following condition on the derivatives of y holds:*

$$y^{(m)}(0) = y^{(m+1)}(0) = \ldots = y^{(2m-1)}(0) = 0, \quad m \ge 1. \tag{4.3}$$

Then the corresponding iterated collocation approximation u^I given by (4.1) satisfies

$$\max_{t_n \in \bar{Z}_N} |e^I(t_n)| = O(h^{2m}), \quad (\text{as } h \downarrow 0,\ Nh = T) \tag{4.4}$$

where $e^I := y - u^I$.

Proof. By (1.3) and (4.1), the iterated collocation error e^I satisfies

$$e^I(t) = \int_0^t p(t, s)e(s)\mathrm{d}s, \quad t \in [0, T]. \tag{4.5}$$

In the previous section it was shown that the collocation error e satisfies:

$$e(t) = \delta(t) + \int_0^t R(t, s)\delta(s)\mathrm{d}s, \quad t \in [0, T], \tag{4.6}$$

where $R(t, s) = s^{\mu-2}/t^{\mu-1}$ and δ is a suitable function which is zero at the collocation points. Using the above formula for e in (4.5) and making use of the equality

$$R(t, s) = p(t, s) + \int_s^t p(t, \tau)R(\tau, s)\mathrm{d}\tau, \tag{4.7}$$

gives

$$e^I(t) = \int_0^t R(t, s)\delta(s)\mathrm{d}s, \quad t \in [0, T]. \tag{4.8}$$

Setting $t = t_n$, $1 \le n \le N$, in (4.8) we obtain

$$e^I(t_n) = \sum_{i=0}^{n-1} h \int_0^1 \frac{(t_i + \tau h)^{\mu-2}}{(t_n)^{\mu-1}}\delta(t_i + \tau h)\mathrm{d}\tau. \tag{4.9}$$

The rest of the proof is based on approximating each of the integrals on the right-hand side of (4.9) by using adequate quadrature rules, in a similar way to the one used in [11]. For this analysis, the cases $i = 0$ and $i \geq 1$ are treated separately. If $i = 0$, we have

$$h\int_0^1 \frac{(\tau h)^{\mu-2}}{(t_n)^{\mu-1}}\delta(\tau h)\mathrm{d}\tau = h\int_0^1 \left(\frac{\tau h}{t_n}\right)^{\mu-1} \frac{\delta(\tau h)}{\tau h}\mathrm{d}\tau \leq h\int_0^1 \frac{\delta(\tau h)}{\tau h}\mathrm{d}\tau, \tag{4.10}$$

since $(\tau h/t_n)^{\mu-1} \leq 1$. We now show that

$$\int_0^1 \frac{\delta(\tau h)}{\tau h}\mathrm{d}\tau = O(h^{2m-1}). \tag{4.11}$$

From (2.12) we have an expression for the collocation error function on the first subinterval $\sigma_0 = [t_0, t_1]$:

$$e(\tau h) = \sum_{l=1}^{m} e(c_l h)\,\lambda_l(\tau) + h^m \frac{y^{(m)}(\xi h)}{m!}\prod_{j=1}^{m}(\tau - c_j), \quad \tau h \in \sigma_0, \tag{4.12}$$

with $0 < \xi < 1$. An application of Lemma 3.1, with $q = 0$, gives $\max_{1\leq j\leq m} |e_0(c_j h)| = O(h^{2m})$. On the other hand, by Taylor's formula, since $y \in C^{2m}[0, T]$, we have

$$\begin{aligned} y^{(m)}(\xi h) &= y^{(m)}(0) + \xi h\, y^{(m+1)}(0) + \frac{\xi^2 h^2}{2} y^{(m+2)}(0) + \cdots + \frac{\xi^m h^m}{m!}\, y^{(2m)}(\eta) \\ &= \frac{\xi^m h^m}{m!}\, y^{(2m)}(\eta), \quad \eta \in (0, h), \end{aligned} \tag{4.13}$$

where we have taken into account (4.3). Then, using (4.13) in (4.12) yields $e(0) = O(h^{2m})$. Now, if we introduce the variable $\rho = s/t$ in (2.17), we obtain the following relation

$$e(0) = \delta(0) + \lim_{t\to 0}\int_0^1 \frac{(t\,\rho)^{\mu-2}}{t^{\mu-1}}\, t\,\delta(t\rho)\,\mathrm{d}\rho = \delta(0)\frac{\mu}{\mu-1}. \tag{4.14}$$

Therefore, we have $\delta(0) = e(0)\frac{\mu-1}{\mu}$. Since $e(0) = O(h^{2m})$ then also $\delta(0) = O(h^{2m})$ and (4.11) follows. This gives a contribution of order $O(h^{2m})$ in (4.9) when $i = 0$.

If $i \geq 1$ in (4.9) we first note that

$$\int_0^1 \frac{(t_i+\tau h)^{\mu-2}}{(t_n)^{\mu-1}}\delta(t_i+\tau h)\mathrm{d}\tau = \int_0^1 \frac{(t_i+\tau h)^{\mu-1}}{(t_n)^{\mu-1}}\frac{1}{(t_i+\tau h)}\delta(t_i+\tau h)\mathrm{d}\tau \leq \int_0^1 \frac{\delta(t_i+\tau h)}{t_i+\tau h}\mathrm{d}\tau.$$

Using an m-point Gauss quadrature rule based on the abscissas t_{il}, with remainder E_{ni}, for each integral, we obtain

$$\int_0^1 \frac{\delta(t_i+\tau h)}{t_i+\tau h}\mathrm{d}\tau = \sum_{l=1}^{m} b_l \frac{\delta(t_{il})}{t_{il}} + E_{ni} = O(h^{2m}), \tag{4.15}$$

where we have used the fact that $\delta(t_{il}) = 0$. Since $g \in C^{2m}[0, T]$, then δ will be $(2m)$-times continuously differentiable on each subinterval (cf. (2.15)). Therefore the error term satisfies $|E_{ni}| = O(h^{2m})$ and we thus get from (4.9), summing up all the contributions,

$$\begin{aligned} |e^I(t_n)| &\leq O(h^{2m}) + h\sum_{i=1}^{n-1}|E_{ni}| \\ &\leq O(h^{2m}) + NhO(h^{2m}), \quad n = 1, \ldots, N. \end{aligned} \tag{4.16}$$

Then, for some constant C independent of h, we get

$$|e^I(t_n)| \leq Ch^{2m}, n = 1, \ldots, N,$$

which gives the desired result. ◇

Corollary 4.1. *If the solution y of Eq. (1.3) belongs to the space $V^0_{2m,\beta}[0, T]$, for some $0 \leq \beta < \mu - 1$, then the result (4.4) stated in Theorem 4.1 is valid: we obtain discrete superconvergence of order $2m$ with the iterated collocation, associated with collocation in $S^{-1}_{m-1}(Z_N)$ based on the Gauss points as collocation parameters.*

Proof. Since $y \in V^0_{2m,\beta}[0, T]$ then the solution $\bar{y}(t) = t^\beta y(t)$ of the transformed equation (3.4) belongs to $V^0_{2m-q}[0, T]$. An application of Theorem 4.1 to Eq. (3.4) yields order $2m$ for the iterated solution associated with the collocation solution. The same is true for the equivalent equation (1.3), by taking into account relations (3.3). ◇

Table 1
Linear collocation and iterated collocation for Example 5.1. The error norms $\max|\bar{y}(t_n) - \bar{u}(t_n)|$ and $\max|\bar{y}(t_n) - \bar{u}^{\mathrm{I}}(t_n)|$

h	Collocation/Gauss	Order	Collocation/Radau *II*	Order	Iterated collocation	Order
0.05	1.3D − 2		1.8D − 5		1.0D − 6	
0.025	3.2D − 3	2.0	2.1D − 6	3.1	5.5D − 8	4.2
0.0125	8.0D − 4	2.0	2.5D − 7	3.1	3.0D − 9	4.2
0.00625	2.0D − 4	2.0	3.0D − 8	3.0	1.7D − 10	4.1
0.003125	5.0D − 5	2.0	3.7D − 9	3.0	9.9D − 12	4.1

Table 2
Quadratic collocation and iterated collocation for Example 5.2. The error norms $\max|\bar{y}(t_n) - \bar{u}(t_n)|$ and $\max|\bar{y}(t_n) - \bar{u}^{\mathrm{I}}(t_n)|$

h	Collocation/Gauss	Order	Collocation/Radau *II*	Order	Iterated collocation	Order
0.05	2.8D − 3		3.6D − 8		3.2D − 9	
0.025	3.5D − 4	3.0	1.0D − 9	5.1	4.7D − 11	6.1
0.0125	4.5D − 5	3.0	3.0D − 11	5.1	7.0D − 13	6.0
0.00625	5.6D − 6	3.0	9.3D − 13	5.0	1.1D − 14	6.0
0.003125	7.0D − 7	3.0	2.9D − 14	5.0	1.6D − 16	6.0

Table 3
Cubic collocation and iterated collocation applied to Example 5.3. The error norms $\max|y(t_n) - u(t_n)|$ and $\max|y(t_n) - u^{\mathrm{I}}(t_n)|$

h	Collocation/Gauss	Order	Collocation/Radau *II*	Order	Iterated collocation	Order
0.1	4.3D − 3		9.3D − 10		1.2D − 10	
0.05	2.8D − 4	3.9	6.5D − 12	7.2	4.5D − 13	8.1
0.025	1.8D − 5	4.0	4.8D − 14	7.1	1.7D − 15	8.1
0.0125	1.1D − 6	4.0	3.7D − 16	7.0	6.3D − 18	8.1

Table 4
Iterated collocation associated with quadratic and cubic collocation. The error norms $\max|y(t_n) - u^{\mathrm{I}}(t_n)|$ for several examples

h	*Quadratic* Example 5.4	Order	*Quadratic* Example 5.6	Order	*Cubic* Example 5.5	Order
0.05	1.1D − 5		5.2D − 8		1.6D − 8	
0.025	1.4D − 6	3.0	1.6D − 9	5.0	1.0D − 9	4.0
0.0125	1.8D − 7	3.0	5.0D − 11	5.0	6.3D − 11	4.0
0.00625	2.2D − 8	3.0	1.6D − 12	5.0	4.0D − 12	4.0
0.003125	2.8D − 9	3.0	4.9D − 14	5.0	2.5D − 13	4.0

The importance of (4.3) for superconvergence $\max|y(t_n) - u^{\mathrm{I}}(t_n)|$.

5. Numerical results

We have considered the numerical solution of equation (1.3), with various choices of $g(t)$ and, consequently, of $y(t)$, for $t \in [0, 2]$. We note that for a forcing function $g(t) = t^{\gamma}(\mu + \gamma - 1)/(\mu + \gamma)$ Eq. (1.3) has the exact solution $y(t) = t^{\gamma}$.

Example 5.1. Let $\mu = 5.9$ and g be such that $y(t) = t^{-0.5}$.

Example 5.2. Let $\mu = 4.5$ and g be such that $y(t) = t^3 + t^{3.5}$.

Example 5.3. Let $\mu = 1.5$ and g be such that $y(t) = t^8 + t^{8.5}$.

Example 5.4. Let $\mu = 1.5$ and g be such that $y(t) = t^3$.

Example 5.5. Let $\mu = 1.5$ and g be such that $y(t) = t^4$.

Example 5.6. Let $\mu = 1.5$ and g be such that $y(t) = t^5 + t^{6.5}$.

Tables 1–4 illustrate the performance of the collocation and iterated collocation methods defined by (2.8) and (4.2), respectively, applied to (1.3) or to the transformed Eq. (3.4), when appropriate. When we use the transformed equation, in the tables we have denoted $\bar{u}$ as the collocation solution associated with $\bar{y}$.

Starting with Example 5.1, the transformed equation (3.4) was considered with $\beta = 4.7$, so that the new solution $\bar{y}(t) = t^{\beta}y(t) = t^{4.2}$ belongs to the space $V_4^0[0, 2]$. In Table 1 we show the error norms at the mesh points obtained by linear collocation, that is, in the space $S_{m-1}^{(-1)}(Z_N)$, with $m = 2$. We have used as collocation parameters the Gauss points $c_1 = (3 - \sqrt{3})/6, c_2 = (3 + \sqrt{3})/6$ (second column of the table) and also the Radau II points $c_1 = 1/3, c_2 = 1$ (fourth column). In the first case, the orders obtained confirm the theoretical result, that is, convergence of order $m = 2$ (cf. (2.9)), also showing that no superconvergence occurs at the mesh points. The results with the Radau II points indicate (local) superconvergence of order $2m - 1 = 3$, thus in agreement with Lemma 3.3. The sixth column shows the performance of iterated collocation, associated with collocation in $S_1^{-1}(Z_N)$ based on the Gauss points as collocation parameters. The results indicate local superconvergence (on $\overline{Z}_N$) of order $2m = 4$, confirming the prediction of Corollary 4.1. Here, (4.3) holds for $\bar{y}$, since $\bar{y}''(0) = \bar{y}'''(0) = 0$.

In Table 2 we have considered collocation and the corresponding iterated collocation in $S_{m-1}^{-1}(Z_N)$, with $m = 3$, for Example 5.2. We have chosen $\beta = 3$ in (3.3) and (3.4) which implies that the new solution $\bar{y}(t) = t^6 + t^{6.5}$ belongs to the space $V_6^0[0, T]$. Collocation was applied to Eq. (3.4) and the error norms at the mesh points displayed on the second column confirm the third order of convergence. In accordance with Lemma 3.3, local superconvergence of order $2m - 1 = 5$ was obtained with the Radau II points ($c_1 = (4 - \sqrt{6})/10$, $c_2 = (4 + \sqrt{6})/10$, $c_3 = 1$) as collocation parameters. Finally, the results with iterated collocation based on the Gauss points ($c_1 = (5 - \sqrt{15})/10$, $c_2 = 1/2$, $c_3 = (5 + \sqrt{15})/10$) gave order $2m = 6$ at the mesh points, as expected from Corollary 4.1. We note that $\bar{y}$ satisfies (4.3), since $\bar{y}'''(0) = \bar{y}^{iv}(0) = \bar{y}^{v}(0) = 0$.

In the case of Example 5.3, cubic collocation and iterated collocation were applied directly to the original equation. In Table 3 the results of the second column confirm the fourth (global) order (cf. (2.9)) and also show that no superconvergence is attained at the grid points using the Gauss points as collocation points. However, since condition (3.1) is satisfied with $m = 4$ and $q = 1$, then order $2m - 1 = 7$ is obtained at the grid points with the Radau II points. Again, a remarkable improvement (order $2m = 8$) over collocation is achieved with iterated collocation, as predicted by (4.4).

In Table 4 we illustrate the importance of condition (4.3) for attaining superconvergence with iterated collocation applied to the original equation. We have considered quadratic and cubic collocation, based on the Gauss points, and also their corresponding iterated solutions. The second column of Table 4 shows the errors for the quadratic case, applied to Example 5.4; the results indicate third order of convergence instead of the optimal order $2m = 6$, which is not surprising since $y'''(0) \neq 0$ (cf. (4.3)). The fourth column of Table 4 shows a similar situation with Example 5.6; there $y'''(0) = y^{iv}(0) = 0$ but $y^{v}(0) \neq 0$ and the discrete convergence order drops to 5. Finally, for Example 5.5, the sixth column illustrates the lack of superconvergence with the iterated solution associated with cubic collocation.

6. Conclusions

This work completes the previous studies for the singular Volterra integral of equations with the noncompact kernel $p(t, s) = s^{\mu-1}/t^{\mu}$, in the case $\mu > 1$. In particular, it is shown that piecewise polynomials of degree $m - 1$ yield global convergence of order m. Moreover, under certain assumptions, it is possible to transform the original equation into a new one whose solution satisfies the conditions for local superconvergence. It is also proved that if the Gauss points are employed as collocation parameters, then a convergence order $2m$ is attained at the grid points with the corresponding iterated solution.

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References

[1] P. Baratella, A.P. Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, J. Comput. Appl. Math. 163 (2004) 401–418.
[2] M.A. Bartoshevich, On a heat conduction problem, Inž.- Fiz. Ž. 28 (1975) 340–346.
[3] H. Brunner, The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes, Math. Comp. 45 (1985) 417–437.
[4] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, 2004.
[5] H. Brunner, Nonpolynomial spline collocation for Volterra equations with weakly singular kernels, SIAM J. Numer. Anal. 20 (1983) 1106–1119.
[6] H. Brunner, P.J. van der Houwen, The Numerical Solution of Volterra Equations, North Holland, 1986.
[7] H. Brunner, S. Nørsett, Superconvergence of collocation methods for Volterra and Abel integral equations of the second kind, Numer. Math. 36 (1981) 347–358.
[8] H. Brunner, A. Pedas, G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations, Math. Comp. 68 (1999) 1079–1095.
[9] H. Brunner, H.J.J. te Riele, Volterra-type integral equations of the second kind with nonsmooth solutions: High order methods based on collocation techniques, J. Integral Equations 6 (1984) 187–203.
[10] T. Diogo, N.B. Franco, P. Lima, High order product integration methods for a Volterra integral equation with logarithmic singular kernel, Commun. Pure Appl. Anal. 3 (2004) 217–235.
[11] T. Diogo, P. Lima, Superconvergence properties of collocation methods for a class of weakly singular Volterra integral equations, J. Comput. Appl. Math. (in press), doi:10.1016/j.cam.2007.01.023.
[12] T. Diogo, P. Lima, M. Rebelo, Numerical solution of a nonlinear Abel type Volterra integral equation, Commun. Pure Appl. Anal. 5 (2006) 277–288.
[13] T. Diogo, S. McKee, T. Tang, A Hermite-type collocation method for the solution of an integral equation with a certain weakly singular kernel, IMA J. Numer. Anal. 11 (1991) 595–605.
[14] T. Diogo, S. Mckee, T. Tang, Collocation methods for second-kind Volterra integral equations with weakly singular kernels, Proc. Roy. Soc. Edinburgh 124A (1994) 199–210.
[15] E.A. Galperin, E.J. Kansa, A. Makroglou, S.A. Nelson, Variable transformations in the numerical solution of the second kind Volterra integral equations with continuous and weakly singular kernels; extension to Fredholm integral equations, J. Comput. Appl. Math. 115 (2000) 193–211.

[16] R. Gorenflo, S. Vessella, Abel Integral Equations, A Historico-Bibliographical Survey, IAGA, Firenze, 1984.
[17] W. Han, Existence, uniqueness and smoothness results for second-kind Volterra equations with weakly singular kernels, J. Integral Equations Appl. 6 (1994) 365–384.
[18] P.M. Lima, T. Diogo, An extrapolation method for a Volterra integral equation with weakly singular kernel, Appl. Numer. Math. 24 (1997) 131–148.
[19] J. Norbury, A.M. Stuart, Volterra integral equations and a new Gronwall inequality (Part I: The linear case), Proc. Roy. Soc. Edinburgh 106A (1987) 361–373.
[20] A. Pedas, G. Vainikko, Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations, Computing 73 (2004) 271–293.
[21] H.J.J. te Riele, Collocation methods for weakly singular second-kind Volterra integral equations with non-smooth solution, IMA J. Numer. Anal. 2 (1982) 437–449.
[22] T. Tang, A note on collocation methods for Volterra integro-differential equations with weakly singular kernels, IMA J. Numer. Math. 13 (1993) 93–99.